# Combinatorial Relativity Theory

## *in PhOENIX*

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*Functions-PhOENIX-Combinatorics*

**Today's Ken Quote:**
"I'm taking the Red Pill…"

You have to let it all go Neo. Fear! Doubt! Disbelief! Free Your Mind! –Morpheus

There is no spoon!

Whereas "Reason" measures the Independence of Structures, "Mysticism" measures their Entanglement!

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# Combinatorial Relativity Theory

## I. Abstract

PhOENIX, the Physics of Networks, Entanglements and Information eXchanges, utilizes the mathematical language of Analytic Combinatorics, as presented by *Flajolet* and *Sedgewick (2009)(2013)*, to demonstrate the structures of Special Relativity Theory (SR), Quantum Field Theory (QF) as well as General Relativity Theory (GR) within its framework. In this first of three papers, we establish a landscape on which PhOENIX can be constructed. To that end, we firstly observe a mathematical connection between the Permutation Generating Functions of Even and Odd Cycles in Combinatorics and the Rapidity and Lorentz Gamma Factor of SR. Because the language of Combinatorics only ascribes *relative meaning* to its structures, many apparently different physics processes may be interrelated when they are defined by the same Combinatoric Structures. This is the guiding principle of General Covariance that all foundational modern physics theories are built upon. By virtue of Diffeomorphisms, we find that this foundational General Covariance Principle is equivalent to the structure of Permutations in Combinatorics which are themselves equivalent to the structure of Combinatorial Cycles. And because Cycles are separable into Even and Odd subsets, there exist Even and Odd subsets of Permutations which are precisely the physics forms of the Rapidity and the Lorentz Gamma Factor, the minimal structures required for SR. As such, we demonstrate that the structure of SR is a direct consequence of Even and Odd Cyclic structure in Combinatorics. Because the relationship between the Cyclic Generating Function and the Permutation Generating Function is exponential, we find that the parameters defining Cycles are fixed and are statistically considered Micro-Canonical, while the parameters defining Permutations are variable and are statistically Canonical. Because of this new understanding, we find that after defining our Generating Function variable $z$ as the Relativistic Velocity $\beta$, this core relativity parameter is really a statistical canonical variable, or Bulk variable, that loses relevant meaning on the Micro-Canonical scale where other characteristics associated directly with cyclic structures take precedence. As such in a simple way, the structure of SR is a statistical/bulk (Canonical) consequence of a deeper fixed Cyclic structure (micro-Canonical) that may be regarded as Quantum Mechanical. And in the following papers, we will demonstrate that these Cyclic Structures are the world of PhOENIX!

In Section II: *Outline to Relativity in PhOENIX*, we present the concise logical/mathematical flow to the paper;

In Section III: *Introduction*, we provide an historical/philosophical context to the requirement of General Covariance in any modern physics theory and additionally where and why this requirement can be naturally found in the language of Combinatorics;

In Section IV: *The Combinatorics of Permutations*, we discuss the connection between Permutations and Cycles by virtue of Generating Functions. We also introduce the statistical link between them that is expressed via a Coarse-Graining parameter $z$;

In Section V: *Cycle Operators and Combinatorial Relativity*, we demonstrate that by simply separating the Cycles into Even and Odd additive subgroups, the minimally necessary mathematical structure of SR is made manifest when the Coarse-Graining parameter $z$ is regarded as the relativistic speed $\beta$;

In Section VI: *Random Permutations and Combinatoral Relativity*, we observe that the Conserved Structures in Random Permutation Statistics give rise exactly to the Conservation of Energy in SR. And by virtue of Coefficient Extraction techniques in Combinatorics, we further find a general relationship between the coefficients of the Odd Cycle-Invariant Functional which provides a key insight into spin-induced quantum distortions of spacetime that we explore in our third paper on Quantum Gravity;

Finally in Section VII: *Concluding Remarks*, we summarize this first paper's background understanding so that in the second and third papers we may further construct the mathematical scaffolding of PhOENIX, the Physics Of Entanglements, Networks and Information eXchanges.

## II. Outline to Relativity in PhOENIX

1. General Covariance defines General Relativity

2. Permutations define General Covariance

3. Permutation Generating Functions (GFs) $P(z)$ define Permutations in Combinatorics

4. Cyclic Generating Functions $G_T(z)=\sum_{i=1}^{\infty}\frac{z^i}{i}$, $\widetilde{G}_T(z)=\sum_{i=1}^{\infty}\frac{(-z)^i}{i}$ define Permutation Generating Functions $P_T(z)=e^{G_T(z)}$, $\widetilde{P}_T(z)=e^{\widetilde{G}_T(z)}$, and as *Sedgewick* and *Flajolet* succinctly explain

   "… permutations uniquely decompose into cycles!"[1]

5. Cyclic Generating Functions can be separated into a sum of Even $G_E(z)$ and Odd Cycle $G_O(z)$ Generating Functions

$$G_T(z)=G_E(z)+G_O(z)$$
$$\widetilde{G}_T(z)=G_E(z)-G_O(z)$$

6. Permutation Generating Functions are therefore separable into Even $P_E(z)$ and Odd $P_O(z)$ Permutation Generating Functions

$$P_O(z)=e^{G_O(z)}$$
$$P_E(z)=e^{G_E(z)}$$
$$P_T(z)=P_O(z)P_E(z)=e^{G_O(z)}e^{G_E(z)}=e^{G_O(z)+G_E(z)}=e^{G_T(z)}$$
$$\widetilde{P}_T(z)=\frac{P_E(z)}{P_O(z)}=e^{-G_O(z)}e^{G_E(z)}=e^{-G_O(z)+G_E(z)}=e^{\widetilde{G}_T(z)}$$

7. The Mathematical Structure of Special Relativity are the Even and Odd Permutation GFs

$$P_T(z)=\frac{1}{1-z} \qquad \widetilde{P}_T(z)=\frac{1}{1+z}$$
$$P_O(z)=\sqrt{\frac{1+z}{1-z}} \qquad P_E(z)=\sqrt{\frac{1}{1-z^2}}$$

8. Letting our Continuous variable $z$ represent the relative speed $z=\beta=\frac{v}{c}$ produces the familiar

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[1] Philippe Flajolet and Robert Sedgewick. *Analytic Combinatorics: Chpt.II. Labelled Structures and EGFS: Section 4: Alignments, Structures and Related Structures: Permutations and Cycles*. Cambridge University Press, Copyright 2009, ISBN 978-0-521-89806-5., p. 120.

$$P_T(\beta)=\frac{1}{1-\beta} \qquad \tilde{P}_T(\beta)=\frac{1}{1+\beta}$$

$$P_O(\beta)=\sqrt{\frac{1+\beta}{1-\beta}}=\gamma(1+\beta) \qquad P_E(\beta)=\sqrt{\frac{1}{1-\beta^2}}=\gamma$$

9. Additionally, we define explicit Energy Functionals $g_-(\beta), g_+(\beta), h(\beta)$ because $z=\beta=\frac{v}{c}=\frac{pc}{E}$

$$P_T^{-1}(\beta)=1-\beta=1-\frac{pc}{E}=\frac{E-pc}{E}$$
$$\rightarrow \quad g_-(\beta)=E\,P_T^{-1}(\beta)=E-pc$$
$$\tilde{P}_T^{-1}(\beta)=1+\beta=1+\frac{pc}{E}=\frac{E+pc}{E}$$
$$\rightarrow \quad g_+(\beta)=E\,\tilde{P}_T^{-1}(\beta)=E+pc$$
$$h^2(\beta)=g_+(\beta)g_-(\beta)=(E+pc)(E-pc)=E_0^2$$
$$=E^2\,\tilde{P}_T^{-1}(\beta)P_T^{-1}(\beta)=\frac{E^2}{\tilde{P}_T(\beta)P_T(\beta)}$$
$$\rightarrow \quad h(\beta)=\frac{\pm E}{\sqrt{\tilde{P}_T(\beta)P_T(\beta)}}=\pm E_0$$

where we have expressed an Invariant Energy $E_0=\pm h(\beta)$ in terms of the Total Permutation GFs

10. These Energy Functionals can also be expressed in terms of the Even and Odd Permutation GFs as well as a Rapidity $\phi$

$$P_E^2(\beta)=P_T\,\tilde{P}_T=\gamma^2=\frac{E^2}{E_0^2}=\frac{E^2}{g_+(\beta)g_-(\beta)}=\frac{E^2}{h^2(\beta)}=\frac{1}{1-\beta^2}=\cosh^2(\phi)$$

$$P_O^2(\beta)=\frac{P_T}{\tilde{P}_T}=\gamma^2(1+\beta)^2=\frac{1}{\gamma^2(1-\beta)^2}=\frac{E+pc}{E-pc}=\frac{g_+(\beta)}{g_-(\beta)}=\frac{1+\beta}{1-\beta}=\frac{(E+|\vec{p}c|)^2}{E_0^2}=(\cosh(\phi)+\sinh(\phi))^2$$

$$\beta=\tanh(\phi)$$

## III. Introduction

The Principle of General Covariance was the keystone to Einstein's General Theory of Relativity (GR) and is today a necessary requirement in any description of Quantum Gravity (QG). As Prugovecki states in his discussion of the operational foundations of SR and GR,

> "The perusal of Einstein's (1905, 1916) seminal papers on special and general relativity reveals that Einstein's key epistemological idea – the idea which constituted the cornerstone of the entire ensuing framework, such as Lorentz transformations, pseudo-Riemannian structure, etc. – resided in the premise that spatio-temporal relationships do not have an *a priori* mathematical meaning (as was assumed in the neo-Kantian outlook that had been prevailing until Einstein's times), but rather that such a meaning has to *result* from empirically verifiable properties of operational definitions of such relationships. For example, although the principle of general covariance is stated in textbooks on general relativity (Adler *et al*., 1975, p. 117; Weinberg, 1973, p. 92, etc.) in various forms, that are not exactly mathematically equivalent (Trautman, 1965) but primarily invoke tensorial features of physical laws, in the original form stated by Einstein (1916) its connotation is straightforward and very clear: physical reality does not reside in the coordinates themselves, but rather in the events which those coordinates label! Thus, Einstein (1918) totally concurred with Kretschmann's (1917) comment that *any* physical law dealing with coordinate-dependent quantities can be written in covariant form."[2]

Mathematically, General Covariance is defined by Diffeomorphism-Invariance

> "For a finite set of points, the diffeomorphism group is simply the symmetric group. Similarly, if M is any manifold there is a group extension $0 \rightarrow \mathrm{Diff}_0(M) \rightarrow \mathrm{Diff}(M) \rightarrow \Sigma(\pi_0(M))$. Here $\mathrm{Diff}_0(M)$ is the subgroup of Diff(M) that preserves all the components of M, and $\Sigma(\pi_0(M))$ is the permutation group of the set $\pi_0(M)$ (the components of M). Moreover, the image of the map $\mathrm{Diff}(M) \rightarrow \Sigma(\pi_0(M))$ is the bijection of $\pi_0(M)$ that preserve diffeomorphism classes."[3]

In this way, the Permutation Group which is defined by the Symmetric Group (Cyclic Group) defines the Group of Diffeomorphisms and therefore mathematically fully expresses the Principle of General Covariance. Additionally as Rovelli recounts

> "*Discussion: relation between regularization and background independence.* This result is very important, and deserves a comment. The first key point is that the coordinate space $\vec{x}$ has no physical significance at all. The physical location of things is only location relative to one another, not the location with respect to the coordinates $\vec{x}$. The diffeomorphism-invariant level of the theory implements this essential general-relativistic requirement. The second point is that the excitations of the theory are quantized. This is reflected in the short-scale discreteness, or in **the discrete combinatorial structure of the states**. This is the result of the quantum mechanical properties of

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[2] Eduard Prugovecki (ed), *Stochastic Quantum Mechanics and Quantum Spacetime: A Consistent Unification of Relativity and Quantum Theory Based on Stochastic Spaces, Part II Quantum Spacetime, Chapter 4 Reciprocity Theory and the Geometrization of Stochastic Quantum Mechanics, Section 4.2: The Operational Foundation of Special and General Relativity*", Springer 1984. ISBN: 90-277-1617-X. Kluwer Academic Publishers, Hingham, MA. Copyright 2004, p. 182.

[3] Diffeomorphism - Wikipedia, the free encyclopedia

> the gravitational field. When these two features are combined, there is literally no longer room for diverging short-distance limits."[4]

And although Rovelli is speaking specifically to the requirements of the Hamiltonian Operator, in his first point, he is also more broadly speaking to the meaning and importance of diffeomorphisms in GR; all physical locations are firstly relational and secondly label-invariant. Combinatorics is a mathematics that best describes both of these requirements.[5] Additionally, in his second point, Rovelli describes the combinatorial structure of states, their short-scale discreteness, as a necessary property for additionally eliminating short-distance divergences. We will explore this second necessity in a subsequent treatise on Combinatorial Scattering Theory.

And without regard to hyperbole and too much philosophical discussion, we present our findings for a general audience with a minimal background in Combinatorics; it is our shortest-path approach. As such, it is recommended that readers utilize the references when concepts need further clarification. In this way, there are many additional paths/arguments that also lead to the same inescapable conclusion: a natural language of Measurement is the counting language of Combinatorics, and as a consequence, both the foundational principles and mathematical structures that currently articulate modern physics exist primitively and fundamentally within its algorithmic framework. And although the Generating Functions of Combinatorics provide the appropriate structures (scaffolding), the translation into physics is provided by PhOENIX; it is analogous to the use of Riemann Geometry as the mathematical underpinning to Einstein's General Theory of Relativity. And although we do not explicitly reveal this PhOENIX heart in this first paper, we are obliged to do so in the third.



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[4] Carlo Rovelli, *Quantum Gravity. Chapter 7: Dynamics and Matter. Section 7.1: Hamiltonian Operator*, Cambridge University Press. ISBN: 0 521 83733 2. Cambridge, England. Copyright 2004, pps. 281-2. (green & bold added by Griggs for emphasis)

[5] And, as we have highlighted in green, this is what Rovelli seems to also be hinting at.

## IV. The Combinatorics of Permutations

In *Analytic Combinatorics* as defined by *Flajolet* and *Sedgewick*, the Univariate Exponential Generating Function of Permutations[6,7] is simply

**(IV-1)**

$$P(z)=\frac{1}{1-z}$$

Additionally, this function can be derived from labelled cycles[8]

**(IV-2)**

$$\ln\left(\frac{1}{1-z}\right)=\sum_{i=1}^{\infty}\frac{z^i}{i}=G_T(z)$$

As such, the univariate Permutation Generating Function is directly related to the univariate Cycle (Cyclic) Generating Function via

**(IV-3)**

$$P(z)=\frac{1}{1-z}=e^{\sum_{i=1}^{\infty}\frac{z^i}{i}}=e^{G_T(z)}$$

In this respect, there are two parameters, one continuous and the other discrete, that define the Permutations:

Firstly, the discrete parameter *i* defines the length of the Cycle and acts as a variable describing a micro-canonical ensemble with a fixed-composition.

Secondly, the continuous parameter *z* acts as a variable describing a canonical ensemble (a thermodynamic system) whose fixed composition is generated by the i-Cycles that are in "thermal equilibrium" (on a fixed lattice of v-vertices) with a "Heat Bath" (Energy-Momentum) delineated by a precise "Temperature" (velocity).[9] Additionally, *z* may be regarded as a field parameter that "marks size in the generating function."[10] As a side note, the art of transforming a micro-canonical variable into a canonical variable is sometimes called "*Coarse-Graining*".[11]

More formally,

**(IV-4)**

$$P(z)=P(z,1,2,\ldots,v)=P(z,1)P(z,2)\cdots P(z,v)=e^{z}e^{\frac{z^2}{2}}\cdots e^{\frac{z^v}{v}}=\prod_{i=1}^{v\to\infty}e^{\frac{z^i}{i}}=e^{\sum_{i=1}^{v\to\infty}\frac{z^i}{i}}$$

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[6] Philippe Flajolet and Robert Sedgewick. *Analytic Combinatorics: Chpt.II. Section 1: Labelled Classes: Permutations, Urns and Circular Graphs*. Cambridge University Press, Copyright 2009, ISBN 978-0-521-89806-5., pps. 98-99.

[7] Philippe Flajolet and Robert Sedgewick. *Analytic Combinatorics: Chpt.II1. Section 4: Inherited Parameters and Exponential MGFS*. Cambridge University Press, Copyright 2009, ISBN 978-0-521-89806-5. Theorem III.4. Inherited parameters and exponential MGFs & *Example III.9: the profile of permutations* (where our Inheritance Parameter /Marker is set to one u=1), p. 175.

[8] Wikipedia. "Random Permutation Statistics: Odd Cycle Invariants: the Recurrence", Random permutation statistics - Wikipedia, the free encyclopedia. Downloaded: January 22, 2014, p. 1.

[9] Wikipedia. "Statistical Mechanics: Three Thermodynamic Ensembles". Statistical mechanics - Wikipedia, the free encyclopedia. Downloaded: January 26, 2014.

[10] Philippe Flajolet and Robert Sedgewick. *Analytic Combinatorics: Chpt.II. Section 1: Labelled Classes:Definition II.2*. Cambridge University Press, Copyright 2009, ISBN 978-0-521-89806-5., p. 98.

[11] Erik Verlinde. "On the Origin of Gravity and the Laws of Newton", Institute for Theoretical Physics, University of Amsterdam, The Netherlands, ArXiv: 1001.0785v1 [hep-th] 6 Jan 2010, pps. 3, 6, 11-13, 16.

And in this way, we can regard the Total Permutation Generating Function as a product of i-Permutation Generating Functions (canonical structures) that individually “mark” a single i-Cycle (micro-canonical structure).

## V. Cycle Operators and Combinatorial Relativity

The Generating Function of Total Cycles $G_T(z)$ is at first glance quite benign, but upon further examination, it becomes a cornerstone to an emergent Combinatorial Relativity Theory. In particular, by separating this operator into *Even and Odd Cycle Operators*[12], the mathematical structure of Relativity Theory is made manifest. Namely,

**(V-1)**

$$G_T(z) = G_{Odd}(z) + G_{Even}(z)$$

And when we allow

**(V-2)**

$$-1 \le z \le +1$$

Then we also have a Total Generating Function whose Even and Odd Operators are Anti-Parallel

**(V-3)**

$$\widetilde{G}_T(z) = G_{Odd}(z) - G_{Even}(z) \quad via \quad G_T(-z) = -\widetilde{G}_T(z)$$

The Generating Function of *Even Cycles* $G_{Even}(z)$ becomes

**(V-4)**

$$G_{Even}(z) = \sum_{i=1}^{\infty}\left(\frac{z^{2i}}{2i}\right) = \frac{1}{2}\ln\left(\frac{1}{1-z^2}\right)$$

and the corresponding Generating Function of *Odd Cycles* $G_{Odd}(z)$ becomes

**(V-5)**

$$G_{Odd}(z) = \sum_{i=1}^{\infty}\left(\frac{z^{2i-1}}{2i-1}\right) = \ln\left(\frac{1}{1-z}\right) - \frac{1}{2}\ln\left(\frac{1}{1-z^2}\right) = \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right)$$

These Cycle Operators are mathematically identical to the core functions of SR. In fact, with the following conjecture,

**(V-6)**

$$z = \frac{v}{c} = \beta$$

the Cyclic Generating Functions immediately produce the two essential elements of SR via

**(V-7)**

$$G_{Even}(z) = \frac{1}{2}\ln\left(\frac{1}{1-z^2}\right) = \ln\sqrt{\frac{1}{1-\beta^2}} = \ln(\gamma)$$

$$\gamma = e^{G_{Even}(z)} \equiv Lorentz\ Factor$$

and

[12] Wikipedia. *Analytic Combinatorics: The Cycle Operator* $\mathcal{C}$. http://en.wikipedia.org/wiki/Analytic_combinatorics. Rendered: September 17, 2013.

**(V-8)**

$$G_{Odd}(z) = \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right) = \frac{1}{2}\ln\left(\frac{1+\beta}{1-\beta}\right) = \ln\sqrt{\frac{1+\beta}{1-\beta}} = \phi$$
$$G_{Odd}(z) = \phi \equiv Rapidity$$

In this way, the exponentiated *Even Cycle Operator* is the famed *Lorentz Factor γ*, while the *Odd Cycle Operator* is the equally famed *Rapidity φ* or hyperbolic angle. In fact, we can demonstrate an interrelationship between these parameters by expressing them as Hyperbolic Functions via

**(V-9)**

$$\gamma = e^{G_{Even}(z)} = \cosh(\phi) = \frac{e^{G_{Odd}(z)} + e^{-G_{Odd}(z)}}{2}$$
$$\beta\gamma = z\gamma = z\,e^{G_{Even}(z)} = \sinh(\phi) = \frac{e^{G_{Odd}(z)} - e^{-G_{Odd}(z)}}{2}$$
$$\beta = z = \tanh(\phi) = \frac{e^{G_{Odd}(z)} - e^{-G_{Odd}(z)}}{e^{G_{Odd}(z)} + e^{-G_{Odd}(z)}} = \frac{e^{G_{Odd}(z)} - e^{-G_{Odd}(z)}}{2\,e^{G_{Even}(z)}} = \frac{e^{G_{Odd}(z)-G_{Even}(z)} - e^{-G_{Odd}(z)-G_{Even}(z)}}{2}$$
$$= \frac{e^{-\tilde{G}_T(z)} - e^{-G_T(z)}}{2}$$

Using this, they can additionally and more easily be defined in terms of Masses, Energies, and Momenta, the standard ingredients for the wonderful recipe called SR via

**(V-10)**

$$Standard-Definitions:$$
$$E_0 = m_0c^2 \equiv Invariant\ (Rest)\ Energy \quad , \quad m_0 \equiv Invariant\ (Rest)\ mass$$
$$E = mc^2 \equiv Relativistic\ Energy \quad , \quad m \equiv Relativistic\ mass = m_0\,\gamma$$
$$\vec{p} = m\vec{v} \equiv Relativistic\ 3-Momentum$$
$$\frac{E}{E_0} = \cosh(\phi) = \gamma = \frac{e^{\phi} + e^{-\phi}}{2} \quad , \quad \frac{|\vec{p}c|}{E_0} = \sinh(\phi) = \beta\gamma = \frac{e^{\phi} - e^{-\phi}}{2}$$
$$\frac{|\vec{p}c|}{E} = \frac{\sinh(\phi)}{\cosh(\phi)} = \tanh(\phi) = \beta = \frac{e^{\phi} - e^{-\phi}}{e^{\phi} + e^{-\phi}}$$

**(V-11)**

$$Additional\ \ Standard-Definitions:$$
$$e^{\phi} = \cosh(\phi) + \sinh(\phi) \quad , \quad e^{-\phi} = \cosh(\phi) - \sinh(\phi)$$
$$e^{\phi}e^{-\phi} = 1 = (\cosh(\phi) + \sinh(\phi))(\cosh(\phi) - \sinh(\phi)) = \cosh^2(\phi) - \sinh^2(\phi)$$
$$= \frac{(E + |\vec{p}c|)(E - |\vec{p}c|)}{E_0^2} = \frac{\left(E^2 - |\vec{p}c|^2\right)}{E_0^2}$$
$$= \gamma^2(1+\beta)(1-\beta) = \gamma^2\left(1-\beta^2\right) = \gamma^2\left(1-\tanh^2(\phi)\right) = \gamma^2\left(sech^2(\phi)\right) = \gamma^2\left(\cosh^{-2}(\phi)\right)$$
$$= 1$$

$$as\ such \quad E_0^2 = E^2 - |\vec{p}c|^2 \quad and \quad \gamma^2 = \frac{1}{1-\beta^2}$$

**(V-12)**

$$
\begin{aligned}
&Cycle - Definitions: \\
&G_T(z) = G_{Odd}(z) + G_{Even}(z) \\
&\phi = G_{Odd}(z) = \frac{1}{2}\ln\left(\frac{E + |\vec{p}c|}{E - |\vec{p}c|}\right) \quad , \\
&\gamma = e^{G_{Even}(z)} \to G_{Even}(z) = \ln\sqrt{\frac{1}{1-\beta^2}} = \frac{1}{2}\ln\left(\frac{E^2}{E^2 - |\vec{p}c|^2}\right) = \ln\left(\frac{E}{E_0}\right) \\
&e^{\phi} = \cosh(\phi) + \sinh(\phi) = \frac{(E + |\vec{p}c|)}{E_0} = \gamma(1+\beta) = e^{G_{Even}(z)}(1+z) = e^{G_{Odd}(z)} \quad , \\
&e^{-\phi} = \cosh(\phi) - \sinh(\phi) = \frac{(E - |\vec{p}c|)}{E_0} = \gamma(1-\beta) = e^{G_{Even}(z)}(1-z) = e^{-G_{Odd}(z)}
\end{aligned}
$$

Now that we know what the Odd Cycle and Even Cycle Operators denote separately, what does the *Total Cyclic Generating Function* represent in SR? To that end,

**(V-13)**

$$
\begin{aligned}
&G_{Odd}(z) = \frac{1}{2}\ln\left(\frac{E + |\vec{p}c|}{E - |\vec{p}c|}\right) \quad , \quad G_{Even}(z) = \frac{1}{2}\ln\left(\frac{E^2}{E^2 - |\vec{p}c|^2}\right) \\
&G_T(z) = G_{Odd}(z) + G_{Even}(z) \\
&G_T(z) = \frac{1}{2}\ln\left(\frac{E + |\vec{p}c|}{E - |\vec{p}c|}\right) + \frac{1}{2}\ln\left(\frac{E^2}{E^2 - |\vec{p}c|^2}\right) = \frac{1}{2}\left(\ln\left(\frac{E + |\vec{p}c|}{E - |\vec{p}c|}\right) + \ln\left(\frac{E^2}{E^2 - |\vec{p}c|^2}\right)\right) \\
&using \quad \ln(ab) = \ln(a) + \ln(b) \quad , \quad \ln\left(a/b\right) = \ln(a) - \ln(b) \\
&G_T(z) = \frac{1}{2}\ln\left(\left(\frac{E + |\vec{p}c|}{E - |\vec{p}c|}\right)\left(\frac{E^2}{E^2 - |\vec{p}c|^2}\right)\right) = \frac{1}{2}\ln\left(\left(\frac{E + |\vec{p}c|}{E - |\vec{p}c|}\right)\left(\frac{E^2}{(E - |\vec{p}c|)(E + |\vec{p}c|)}\right)\right) \\
&= \frac{1}{2}\ln\left(\left(\frac{1}{E - |\vec{p}c|}\right)\left(\frac{E^2}{(E - |\vec{p}c|)}\right)\right) = \frac{1}{2}\ln\left(\left(\frac{E^2}{(E - |\vec{p}c|)^2}\right)\right) = \frac{1}{2}\ln\left(\left(\frac{E}{E - |\vec{p}c|}\right)^2\right) \\
&= \frac{2}{2}\ln\left(\frac{E}{E - |\vec{p}c|}\right) = \ln\left(\frac{E}{E - |\vec{p}c|}\right) = \ln\left(\frac{1}{1 - |\vec{p}c|/E}\right) = \ln\left(\frac{1}{1-\beta}\right) \\
&G_T(z) = \ln\left(\frac{1}{1-\beta}\right) \\
&\frac{1}{1-\beta} = e^{G_T(z)}
\end{aligned}
$$

In fact, we have just derived the *Total Permutation Generating Function*![13] This is of course, what we should have expected for the Total Cyclic Generating Function…Lol.

[13] Philippe Flajolet and Robert Sedgewick. *Analytic Combinatorics: Chpt.II1.Section 4: Inherited Parameters and Exponential MGFS*. Cambridge University Press, Copyright 2009, ISBN 978-0-521-89806-5. Theorem III.2. Inherited parameters and exponential MGFs, p. 175.

## VI. Random Permutations and Combinatoral Relativity

In their exposé on *Random Permutation Statistics*[14], *Flajolet* and *Sedgewick* are cited once again for their use of Generating Functions in their *Analytic Combinatorial* exploration of *Permutations*. In particular, they present an Invariant Function of Even Cycles[15] which is independent of the Odd Cycles

> "The term odd cycle invariant simply means that membership in the respective combinatorial class is independent of the size and number of odd cycles occurring in the permutation. In fact, we can prove that all odd cycle invariants obey a simple recurrence, which we will derive."[16]

Because this nomenclature can be confusing, we will simply call them *Cycle-Invariant Functions*. Thus,

**(VI-1)**

$$g(z) = h(z)\sqrt{\frac{1+z}{1-z}}$$

$$h(+z) = h(-z), \quad \textit{Even Cycle Function}$$

$$\textit{such that}$$

$$\frac{1}{1+z}g(+z) = h(z)\frac{1}{\sqrt{1-z^2}} = \frac{1}{1-z}g(-z)$$

$$h(z) = \frac{\sqrt{1-z^2}}{1+z}g(+z) = \sqrt{\frac{1-z^2}{(1+z)^2}}g(+z) = \sqrt{\frac{1-z}{1+z}}g(+z) = \frac{g(-z)g(+z)}{h(-z)} = \frac{g(-z)g(+z)}{h(z)}$$

$$h^2(z) = g(+z)g(-z)$$

$$g(+z) = \frac{1+z}{1-z}g(-z) \quad \rightarrow \quad g(+z) \neq g(-z)\, \textit{unless } z = 0$$

$$\textit{Thus } g(z) = \textit{Odd Cycle Function}$$

By recalling the following associations of Eqn. (V-6),

**(VI-2)**

$$z \equiv \beta = \frac{v}{c} \quad \textit{Continuous Field Parameter}$$

$$g(+\beta) = E + |\vec{p}c| \quad , \quad g(-\beta) = E - |\vec{p}c| \quad \textit{Odd Generating Function}$$

$$h^2(\pm\beta) = g(+\beta)g(-\beta) = (E + |\vec{p}c|)(E - |\vec{p}c|) = E_0^2 \quad \textit{Even Generating Function}$$

$$\textit{where,}$$

$$\underset{\textit{Relativistic Energy}}{E(\pm\beta) = mc^2\beta^2} \quad , \quad \underset{\textit{Relativistic Momentum}}{|\vec{p}(\pm\beta)c| = \pm mc^2\beta} \quad , \quad \underset{\textit{Invariant Energy}}{E_0^2 = m_0c^2 = E^2 - |\vec{p}c|^2}$$

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[14] Wikipedia. "Random Permutation Statistics: The Fundamental Relation", Random permutation statistics - Wikipedia, the free encyclopedia. Downloaded: January 22, 2014.

[15] Wikipedia. "Random Permutation Statistics: Odd Cycle Invariants: The Recurrence", Downloaded: January 22, 2014, p. 14.

[16] Wikipedia. "Random Permutation Statistics: Odd Cycle Invariants: The Recurrence", Downloaded: January 22, 2014, p. 13.

We find that these *Cycle-Invariant Functions* are a combinatorial expression of the *Conservation of Energy-Momentum in Relativity Theory*; whereby the Even Cyclic-Invariant Function defines the *Invariant Energy* $E_0$ (Rest Energy) of the field. Thus,

**(VI-3)**

$$g(\pm\beta)=h(\beta)\sqrt{\frac{1\pm\beta}{1\mp\beta}} \quad\leftrightarrow\quad E\pm|\vec{p}c|=E_0\sqrt{\frac{1\pm\beta}{1\mp\beta}}$$

$$\frac{1}{1+\beta}g(+\beta)=h(\beta)\frac{1}{\sqrt{1-\beta^2}}=\frac{1}{1-\beta}g(-\beta) \quad\leftrightarrow$$

$$\frac{E+|\vec{p}c|}{1+\beta}=\frac{E_0}{\sqrt{1-\beta^2}}=\frac{E-|\vec{p}c|}{1-\beta}$$

$$\left(E+|\vec{p}c|\right)\sqrt{\frac{1-\beta}{1+\beta}}=E_0=\sqrt{\frac{1+\beta}{1-\beta}}\left(E-|\vec{p}c|\right)$$

$$\frac{E+|\vec{p}c|}{1+\beta}=\gamma E_0=\frac{E-|\vec{p}c|}{1-\beta}$$

$$\frac{E+|\vec{p}c|}{\gamma(1+\beta)}=E_0=\frac{E-|\vec{p}c|}{\gamma(1-\beta)}$$

We can extract the Coefficients of our Odd Generating Function in the familiar/standard way,[17,18] where we notice that our function is the simple exponential having very well-known inverse-factorial coefficients

**(VI-4)**

$$g_n\equiv[z^n]g(\pm\beta)=[z^n]E_0\left(E\pm|\vec{p}c|\right)=E_0[z^n]\left(\cosh(\phi)\pm\sinh(\phi)\right)$$

$$=E_0[z^n]e^{\pm\phi}=E_0\frac{(\pm1)^n}{n!}$$

and in this way, we obtain the following recurrence relation

**(VI-5)**

$$g_n=E_0\frac{(\pm1)^n}{n!}=E_0\frac{(\pm1)^n}{n!}=E_0\frac{\pm1}{n}\frac{(\pm1)^{n-1}}{(n-1)!}=\frac{\pm1}{n}\left(E_0\frac{(\pm1)^{n-1}}{(n-1)!}\right)=\pm\frac{1}{n}g_{n-1}$$

$$g_{n-1}=\pm n\, g_n$$

And by repeating this technique, we can generalize the recurrence relation via

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[17] Philippe Flajolet and Robert Sedgewick. *Analytic Combinatorics: Chpt. II. Section 1: Labelled Classes*. Cambridge University Press, Copyright 2009, ISBN 978-0-521-89806-5, p. 98.

[18] Philippe Flajolet and Robert Sedgewick. *Analytic Combinatorics: Chpt.II1.Section 4: Inherited Parameters and Exponential MGFS*. Cambridge University Press, Copyright 2009, ISBN 978-0-521-89806-5. Theorem III.2. Inherited parameters and exponential MGFs, p. 175.

**(VI-6)**

$$g_n = \pm\frac{1}{n}g_{n-1} = \frac{\pm 1}{n}\left(\frac{\pm 1}{(n-1)}E_0\frac{(\pm 1)^{n-2}}{(n-2)!}\right) = \frac{1}{n(n-1)}g_{n-2} = \frac{\pm 1}{n(n-1)(n-2)}g_{n-3} = \frac{(\pm 1)^p}{n(n-1)(n-2)\cdots(n-p+1)}g_{n-p}$$

$$g_n = (\pm 1)^p\frac{(n-p)!}{n!}g_{n-p}$$

$$relabelling \quad m = n-p \quad then$$

$$g_n = (\pm 1)^{n-m}\frac{m!}{n!}g_m \quad , \quad 0 \le (m \le n) \in \Im$$

In this way, there is a general relationship between the coefficients of the Odd Cycle-Invariant Functional. In fact, we will extensively explore this relationship in our third paper on Quantum Gravity; it arises to explain the relationship between a particle's instrinsic quantum spin (Fermionic or Bosonic) and quantum distortions in the geometry of spacetime.

To close this section, we provide an additional check that these current associations are consistent with our previous assertions of Eqns. (V-9), (V-10). Recall that

**(VI-7)**

$$Let \quad z = f(z) = \beta$$

$$G_{Odd}(z) = \frac{1}{2}\log\frac{1+f(z)}{1-f(z)} = \frac{1}{2}\log\frac{1+\beta}{1-\beta} = \log\sqrt{\frac{1+\beta}{1-\beta}} = \phi$$

$$G_{Odd}(z) = \phi \equiv Rapidity = Proper\ Time \quad , \quad G_{Even}(z) = \varphi \equiv Proper\ Space$$

$$e^{\pm\varphi} = \gamma = e^{G_{Even}(z)} = \sqrt{\frac{1}{1-\beta^2}} = \sqrt{\frac{E^2}{E^2 - |\vec{p}c|^2}} = \frac{E}{E_0} = \cosh(\phi) = \frac{e^{G_{Odd}(z)} + e^{-G_{Odd}(z)}}{2} \equiv Lorentz\ Factor$$

$$e^{\pm\phi} = \gamma(1\pm\beta) = e^{\pm G_{Odd}(z)} = \sqrt{\frac{1\pm\beta}{1\mp\beta}} = \sqrt{\frac{E\pm|\vec{p}c|}{E\mp|\vec{p}c|}} = \frac{(E\pm|\vec{p}c|)}{E_0} = \cosh(\phi)\pm\sinh(\phi) \equiv Exponentiated\ Rapidity$$

$$\beta\gamma = f(z)\gamma = f(z)e^{G_{Even}(z)} = \frac{|\vec{p}c|}{E_0} = \sinh(\phi) = \frac{e^{G_{Odd}(z)} - e^{-G_{Odd}(z)}}{2}$$

$$\beta = f(z) = \frac{|\vec{p}c|}{E} = \tanh(\phi) = \frac{e^{G_{Odd}(z)} - e^{-G_{Odd}(z)}}{e^{G_{Odd}(z)} + e^{-G_{Odd}(z)}} = \frac{e^{\tilde{G}_T(z)} - e^{-G_T(z)}}{2}$$

$$+G_T(z) = +G_{Odd}(z) + G_{Even}(z) = +\phi + \varphi$$

$$+\tilde{G}_T(z) = +G_{Odd}(z) - G_{Even}(z) = +\phi - \varphi$$

We see that by virtue of

**(VI-8)**

$$e^{\pm G_{Odd}(z)} = \sqrt{\frac{1\pm\beta}{1\mp\beta}} = \sqrt{\frac{1\pm\frac{|\vec{p}c|}{E}}{1\mp\frac{|\vec{p}c|}{E}}} = \sqrt{\frac{\frac{E\pm|\vec{p}c|}{E}}{\frac{E\mp|\vec{p}c|}{E}}} = \sqrt{\frac{E\pm|\vec{p}c|}{E\mp|\vec{p}c|}} = \sqrt{\frac{E\pm|\vec{p}c|}{E\mp|\vec{p}c|}\left(\frac{E\pm|\vec{p}c|}{E\pm|\vec{p}c|}\right)}$$

$$= \sqrt{\frac{\left(E\pm|\vec{p}c|\right)^2}{E^2-|\vec{p}c|^2}} = \sqrt{\frac{\left(E\pm|\vec{p}c|\right)^2}{E_0^2}}$$

$$= \frac{E\pm|\vec{p}c|}{E_0}$$

We have

**(VI-9)**

$$g(\pm z) = h(z)\sqrt{\frac{1+z}{1-z}} = h(z)\frac{E\pm|\vec{p}c|}{E_0} = h(z)e^{\pm G_{Odd}(z)}$$

*by assuming*

$h(z) = E_0$ ,  *Even Function*

*then*

$g(\pm z) = E\pm|\vec{p}c|$,  *Odd Function*

Which demonstrates that our Even Function *h(z)* and Odd Function *g(z)* associations of Eqn.(VI-2) are in fact fully consistent with all of our prior understandings.

## VII. Concluding Remarks

In this way, the Permutation Generating Function that fundamentally defines the General Covariance of General RelativityTheory naturally gives rise to the mathematical structure of Special Relativity Theory. Because Analytic Combinatorics provides a natural language for these structures, we will present in the next (second) paper the appropriate combinatorial generalization to Relativistic Quantum Field Theory or Scattering Theory and demonstrate in our final (third) paper of this introductory series to PhOENIX its seemless generalization to Quantum General Relativity.

If Permutations are our Tree of Knowledge in this Garden of Eden called Combinatorics (the Matrix), then what other trees, vegetation, flora, fauna or animal life also exist? And if there are others, unseen so far by mythology and time, what new discoveries will they reveal and unveil?…

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